Revision Guide for Chapter 10

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Revision Checklist

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I can show my understanding of effects, ideas and relationships by describing and explaining cases involving:

capacitance as the ratio $C = Q / V$ the energy stored on a capacitor: $E = \frac{1}{2}QV$ Revision Notes: capacitance Summary Diagrams: Analogies between charge and water, Energy stored by a capacitor	
decay of charge on a capacitor modelled as an exponential relationship between charge and time, with the rate of removal of charge proportional to the quantity of charge remaining: $\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{Q}{RC}$ Revision Notes: exponential decay processes, differential equation Summary Diagrams: Analogies between charge and water, Exponential decay of charge	
radioactive decay modelled as an exponential relationship between the number of undecayed atoms, with a fixed probability of random decay per atom per unit time $\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N$ Revision Notes: exponential decay processes, differential equation Summary Diagrams: Smoothed out radioactive decay, Radioactive decay used as a clock, Half-life and time constant	
simple harmonic motion of a mass *m* subject to a restoring force $F = -kx$ proportional to the displacement: $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{k}{m}x$ Revision Notes: harmonic oscillator, simple harmonic motion, differential equation Summary Diagrams: A language to describe oscillations, Snapshots of the motion of a simple harmonic oscillator, Graphs of simple harmonic motion, Step by step through the dynamics	
energy $(1/2)kx^2$ stored in a stretched spring changes of kinetic energy $(1/2)mv^2$ and potential energy $(1/2)kx^2$ during simple harmonic motion Revision Notes: simple harmonic motion Summary Diagrams: Elastic energy, Energy flow in an oscillator	
free or forced vibrations (oscillations) of an object damping of oscillations resonance (i.e. when natural frequency of vibration matches the driving frequency) Revision Notes: resonance and damping	

Summary Diagrams: Resonance	

I can use the following words and phrases accurately when describing effects and observations:

for capacitors: half-life, time constant for radioactivity: half-life, decay constant, random, probability Revision Notes: exponential decay processes	
simple harmonic motion, amplitude, frequency, period, free and forced oscillations, resonance Revision Notes: simple harmonic motion, resonance and damping Summary Diagrams: A language to describe oscillations, Resonance	
relationships of the form $dx/dy = -kx$, i.e. where a rate of change is proportional to the amount present Revision Notes: exponential decay processes, differential equation	

I can sketch, plot and interpret graphs of:

radioactive decay against time (plotted both directly and logarithmically) Revision Notes: exponential decay processes Summary Diagrams: Radioactive decay used as a clock, Half-life and time constant	
decay of charge, current or potential difference with time for a capacitor (plotted both directly and logarithmically) Revision Notes: exponential decay processes Summary Diagrams: Analogies between charge and water, Exponential decay of charge	
charge against voltage for a capacitor as both change, and know that the area under the curve gives the corresponding energy change Revision Notes: capacitance Summary Diagrams: Energy stored by a capacitor	
displacement–time, velocity–time and acceleration–time of simple harmonic motion (showing phase differences and damping where appropriate) Revision Notes: simple harmonic motion, sine and cosine functions Summary Diagrams: Graphs of simple harmonic motion	
variation of potential and kinetic energy with time in simple harmonic motion Revision Notes: simple harmonic motion Summary Diagrams: Energy flow in an oscillator	
variation in amplitude of a resonating system as the driving frequency changes Revision Notes: resonance and damping Summary Diagrams: Resonance	

I can make calculations and estimates making use of:

iterative numerical or graphical methods to solve a model of a decay equation iterative numerical or graphical methods to solve a model of simple harmonic motion Revision Notes: exponential decay processes, differential equation Summary Diagrams: Step by step through the dynamics, Rates of change, Comparing models	
data to calculate the time constant $\tau = RC$ of a capacitor circuit data to calculate the activity and half-life of a radioactive source Revision Notes: exponential decay processes Summary Diagrams: Smoothed out radioactive decay, Half-life and time constant	
the relationships for capacitors: $C = Q / V$ $I = \Delta Q / \Delta t$ $E = (1/2)\ QV = (1/2)\ CV^2$ Revision Notes: capacitance Summary Diagrams: Energy stored by a capacitor	
the basic relationship for simple harmonic motion: $\frac{d^2x}{dt^2} = a = -(\frac{k}{m})x$ the relationships $x = A \sin 2\pi ft$ and $x = A \cos 2\pi ft$ for harmonic oscillations the period of simple harmonic motion: $T = 2\pi\sqrt{\frac{m}{k}}$ and the relationship $F = -kx$ Revision Notes: simple harmonic motion Summary Diagrams: A language to describe oscillations, Snapshots of the motion of a simple harmonic oscillator, Graphs of simple harmonic motion, Step by step through the dynamics	
the conservation of energy in undamped simple harmonic motion: $E_{\text{total}} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$ Revision Notes: simple harmonic motion Summary Diagrams: Energy flow in an oscillator	

Revision Notes

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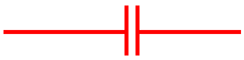
Capacitance

Capacitance is charge separated / potential difference, $C = Q/V$.

The SI unit of capacitance is the farad (symbol F).

Capacitor symbol



One farad is the capacitance of a capacitor that separates a charge of one coulomb when the potential difference across its terminals is one volt. This unit is inconveniently large. Thus capacitance values are often expressed in microfarads (μF) where 1 μF = 10^{-6} F.

Relationships

For a capacitor of capacitance C charged to a potential difference V:

Charge stored $Q = C\,V$.

Energy stored in a charged capacitor $E = \frac{1}{2}\,Q\,V = \frac{1}{2}\,C\,V^2$.

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Exponential decay processes

In an exponential decay process the rate of decrease of a quantity is proportional to the quantity remaining (i.e. the quantity that has not yet decayed).

Capacitor discharge

For capacitor discharge through a fixed resistor, the current I at any time is given by $I = V / R$, where $V = Q / C$. Hence $I = Q / RC$.

Thus the rate of flow of charge from the capacitor is

$$I = \frac{dQ}{dt} = -\frac{Q}{RC}$$

where the minus sign represents the decrease of charge on the capacitor with increasing time.

The solution of this equation is

$$Q = Q_0 e^{-t/RC}.$$

The time constant of the discharge is RC.

Radioactive decay

The disintegration of an unstable nucleus is a random process. The number of nuclei δN that disintegrate in a given short time δt is proportional to the number N present:

$\delta N = -\lambda N \delta t$, where λ is the decay constant. Thus:

$$\frac{\delta N}{\delta t} = -\lambda N.$$

If there are a very large number of nuclei, the model of the differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N$$

can be used. The solution of this equation is

$$N = N_0 \, \mathrm{e}^{-\lambda t}.$$

The time constant is $1 / \lambda$. The half-life is $T_{1/2} = \ln 2 / \lambda$.

Step by step computation
Both kinds of exponential decay can be approximated by a step-by-step numerical computation.

1. Using the present value of the quantity (e.g. of charge or number of nuclei), compute the rate of change.
2. Having chosen a small time interval d*t*, multiply the rate of change by d*t*, to get the change in the quantity in time d*t*.
3. Subtract the change from the present quantity, to get the quantity after the interval d*t*.
4. Go to step 1 and repeat for the next interval d*t*.

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Differential equation

Differential equations describe how physical quantities change, often with time or position.

The rate of change of a physical quantity, y , with time t is written as $\mathrm{d}y/\mathrm{d}t$.

The rate of change of a physical quantity, y , with position x is written as $\mathrm{d}y/\mathrm{d}x$.

A rate of change can itself change. For example, acceleration is the rate of change of velocity, which is itself the rate of change of displacement. In symbols:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)$$

which is usually written

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}.$$

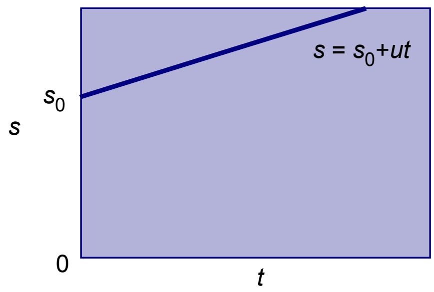
A first-order differential equation is an equation which gives the rate of change of a physical quantity in terms of other quantities. A second-order differential equation specifies the rate of change of the rate of change of a physical quantity.

Some common examples of differential equations in physics are given below.

Constant rate of change
The simplest form of a differential equation is where the rate of change of a physical quantity is constant. This may be written as $\mathrm{d}y/\mathrm{d}t = k$ if the change is with respect to time or $\mathrm{d}y/\mathrm{d}x = k$ if the change is with respect to position.

An example is where a vehicle is moving along a straight line at a constant velocity u . Since its velocity is its rate of change of displacement ds /dt , then ds /dt = u is the differential equation describing the motion. The solution of this equation is $s = s_0 + u\,t$, where s_0 is the initial distance.

Motion at constant velocity, *u*



Second order differential equation

Another simple differential equation is where the second-order derivative of a physical quantity is constant.

For example, the acceleration d^2s/dt^2 (the rate of change of the rate of change of displacement) of a freely falling object (if drag is negligible) is described by the differential equation

$$\frac{d^2s}{dt^2} = -g$$

where g is the acceleration of free fall and the minus sign represents downwards motion when the distance s is positive if measured upwards.

Then

$$s = s_0 + ut - \tfrac{1}{2}gt^2$$

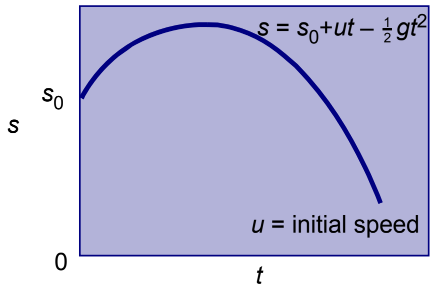
can be seen to be the solution of the differential equation, since differentiating s once gives

$$\frac{ds}{dt} = u - gt$$

and differentiating again gives

$$\frac{d^2s}{dt^2} = -g.$$

Motion at constant acceleration, – *g*



The simple harmonic motion equation

$$\frac{d^2s}{dt^2} = -\omega^2 x$$

represents any situation where the acceleration of an oscillating object is proportional to its displacement from a fixed point. The solution of this equation is

$$s = A\sin(\omega t + \phi)$$

where A is the amplitude of the oscillations and ϕ is the phase angle of the oscillations. If s = 0 when t = 0 , then ϕ = 0 and so

$$s = A\sin(\omega t).$$

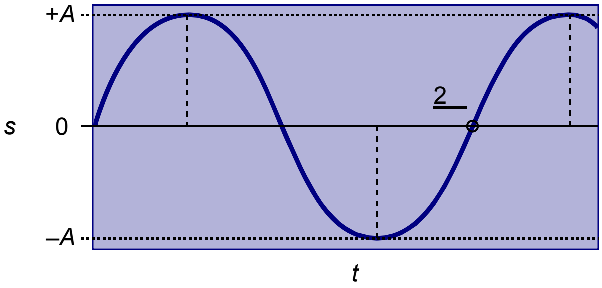
If s = A when t = 0, then $\phi = \pi / 2$ and so

$$s = A\cos(\omega t)$$

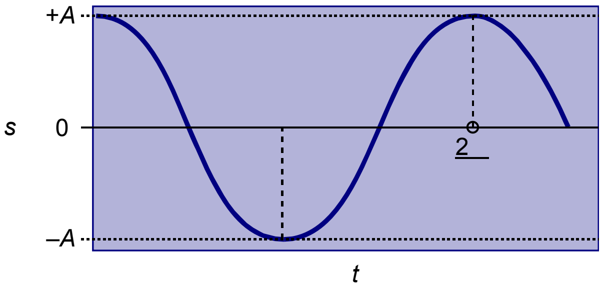
because

$$\sin\left(\omega t + \frac{\pi}{2}\right) = \cos(\omega t).$$

Sinusoidal curves



$s = A \sin \quad t$



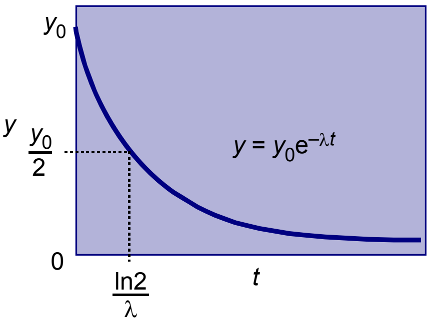
$s = A \cos \quad t$

Exponential decay

The exponential decay equation $dy / dt = -\lambda y$ represents any situation where the rate of decrease of a quantity is in proportion to the quantity itself. The constant λ is referred to as the decay constant. Examples of this equation occur in capacitor discharge, and radioactive decay.

The solution of this differential equation is $y = y_0 e^{-\lambda t}$ where y_0 is the initial value. The half-life of the process is $\ln 2 / \lambda$.

Exponential decrease



Relationships

Differential equations for:

1. Constant speed

$$\frac{ds}{dt} = u.$$

2. Constant acceleration

$$\frac{d^2s}{dt^2} = -g.$$

3. Simple harmonic motion

$$\frac{d^2s}{dt^2} = -\omega^2 x.$$

4. Exponential decay

$$\frac{dy}{dt} = -\lambda y.$$

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Harmonic oscillator

A harmonic oscillator is an object that vibrates at the same frequency regardless of the amplitude of its vibrations. Its motion is referred to as simple harmonic motion.

The acceleration of a harmonic oscillator is proportional to its displacement from the centre of oscillation and is always directed towards the centre of oscillation.

In general, the acceleration $a = -\omega^2 s$, where s is the displacement and ω the angular frequency of the motion = $2\pi / T$, where T is the time period.

Relationships

The displacement of a harmonic oscillator varies sinusoidally with time in accordance with an equation of the form

$$s = A\sin(\omega t + \phi)$$

where A is the amplitude of the oscillations and ϕ is an angle referred to as the phase angle of the motion, taken at time $t = 0$.

Acceleration $a = -\omega^2 s$.

The angular frequency of the motion $\omega = 2\pi / T$.

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Simple harmonic motion

Simple harmonic motion is the oscillating motion of an object in which the acceleration of the object at any instant is proportional to the displacement of the object from equilibrium at that instant, and is always directed towards the centre of oscillation (i.e. the equilibrium position).

The oscillating object is acted on by a restoring force which acts in the opposite direction to the displacement from equilibrium, slowing the object down as it moves away from equilibrium and speeding it up as it moves towards equilibrium.

The acceleration $a = F/m$. For restoring forces that obey Hooke's Law, $F = -ks$ is the restoring force at displacement s. Thus the acceleration is given by:
$a = -(k/m)s$,

The solution of this equation takes the form

$s = A\sin(2\pi ft + \phi)$ where the frequency f is given by $(2\pi f)^2 = k/m$, and ϕ is a phase angle.

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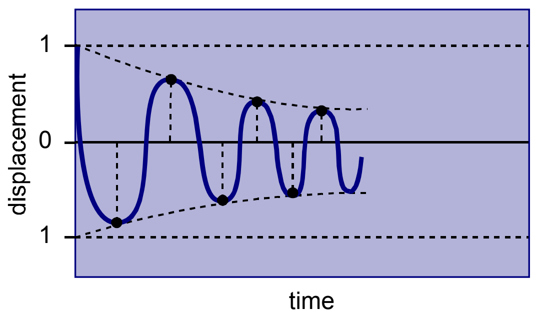
Resonance and damping

In any oscillating system, energy is passed back and forth between parts of the system:

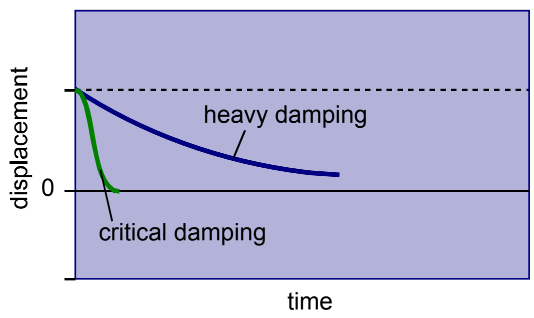
1. If no damping is present, the total energy of an oscillating system is constant. In the mechanical case, this total energy is the sum of its kinetic and potential energy at any instant.
2. If damping is present, the total energy of the system decreases as energy is passed to the surroundings.

If the damping is light, the oscillations gradually die away as the amplitude decreases.

Damped oscillations

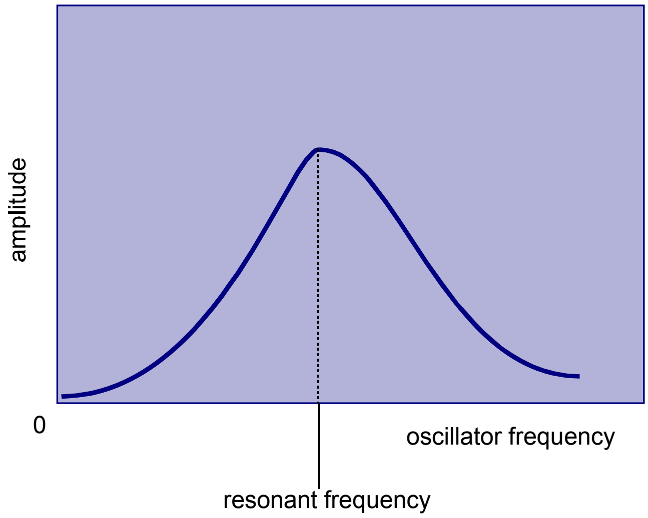


Lightly damped oscillations



Increased damping

Forced oscillations are oscillations produced when a periodic force is applied to an oscillating system. The response of a resonant system depends on the frequency f of the driving force in relation to the system's own natural frequency, f_0. The frequency at which the amplitude is greatest is called the resonant frequency and is equal to f_0 for light damping. The system is then said to be in resonance. The graph below shows a typical response curve.



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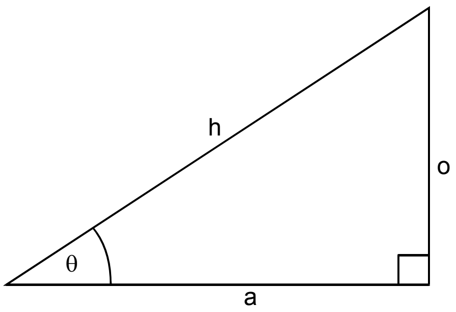
Sine and cosine functions

Sine and cosine functions express an angle in terms of the sides of a right-angled triangle containing the angle.

Sine and cosine functions are very widely used in physics. Their uses include resolving vectors and describing oscillations and waves.

Consider the right-angled triangle shown below.

Trigonometry



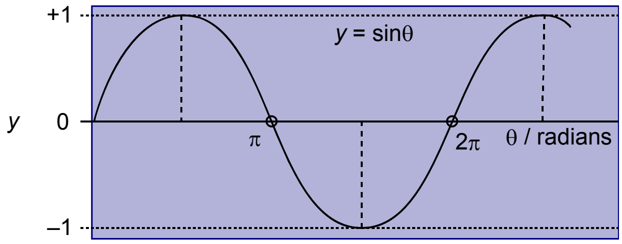
$$\sin\theta = \frac{o}{h}$$

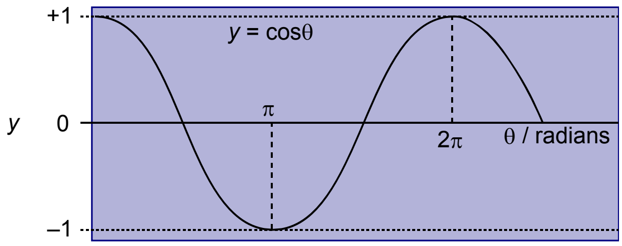
$$\cos\theta = \frac{a}{h}$$

$$\tan\theta = \frac{o}{a}$$

The graphs below show how sin θ and cos θ vary with θ from 0 to 2π radians (= 360°). Note that both functions vary between + 1 and – 1 over 180°, differing only in that the cosine function is 90° out of phase with the sine function. The shape of both curves is the same and is described as **sinusoidal**.

Sinusoidal curves



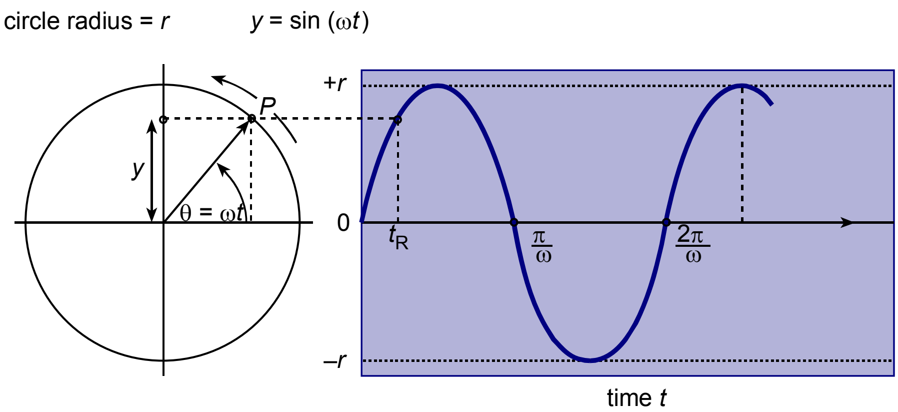


The following values of each function are worth remembering:

	0°	30°	45°	60°	90°	180°
sin θ	0.0	0.5	$1 / \sqrt{2}$	$\sqrt{3} / 2$	1.0	0.0
cos θ	1.0	$\sqrt{3} / 2$	$1 / \sqrt{2}$	0.5	0.0	–1.0

For angle θ less than about 10°, cos θ ≈ 1, and sin θ ≈ tan θ ≈ θ in radians.

Generating a sine curve



Consider a point P moving anticlockwise round a circle of radius r at steady speed, taking time T for one complete rotation, as above. At time t after passing through the +x-axis, the angle between OP and the x-axis, in radians, $\theta = \omega t$ where $\omega = 2\pi / T$. The coordinates of point P are $x = r \cos(\omega t)$ and $y = r \sin(\omega t)$. The curves of sin θ and cos θ against time t occur in simple harmonic motion and alternating current theory.

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Summary Diagrams

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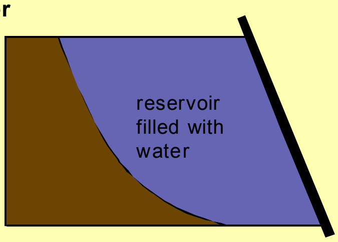
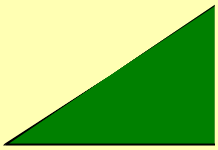
Analogies between charge and water

Storing charge / storing water
Potential difference depends on the quantity stored in both cases.

Water and electric charge

water

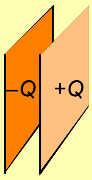


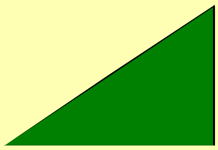


pressure difference increases as amount of water behind dam increases

Electric charge

conducting plates with opposite charges concentrated on them





potential difference V increases as amount of charge separated increases

define capacitance:

$$C = \frac{Q}{V}$$

charge separated per volt

to calculate Q or V:

$$Q = CV$$

$$V = \frac{Q}{C}$$

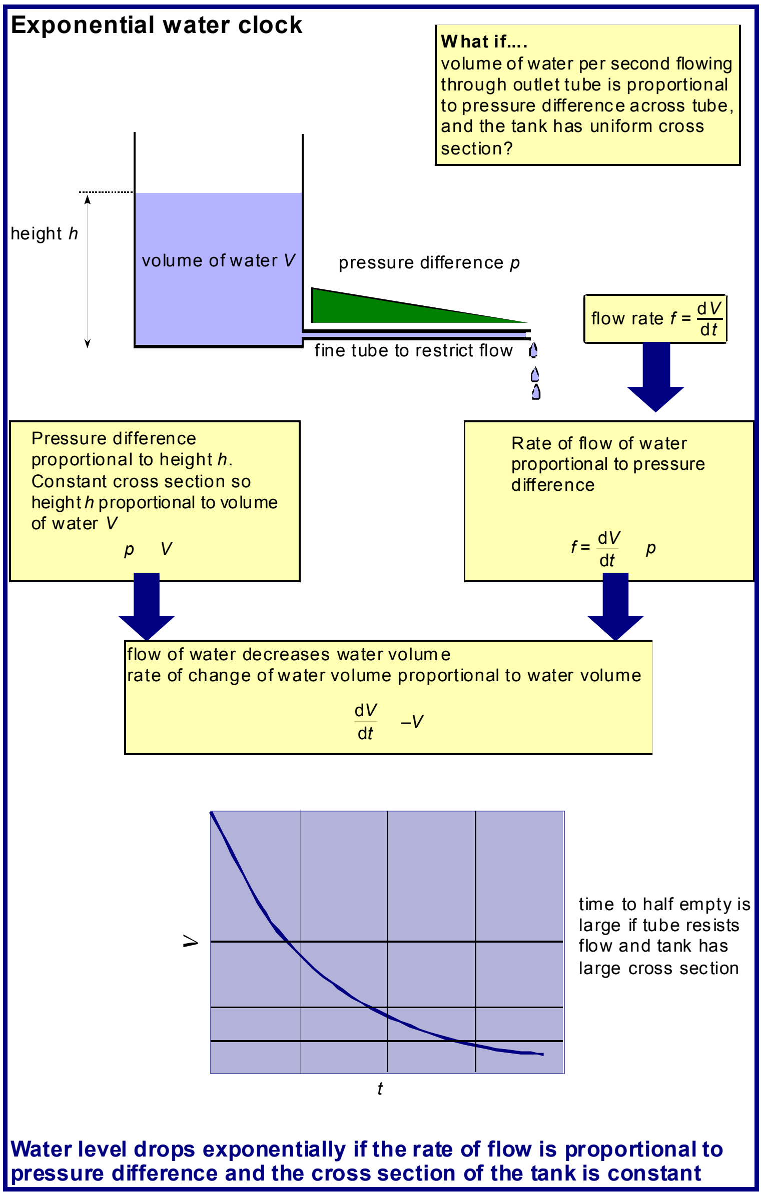
units:

charge Q	coulomb C
potential difference V	volt V
capacitance C	farad F = C V^{-1}

Capacitors keep opposite charges separated. The charge is proportional to the capacitance at a given potential difference

Water running out

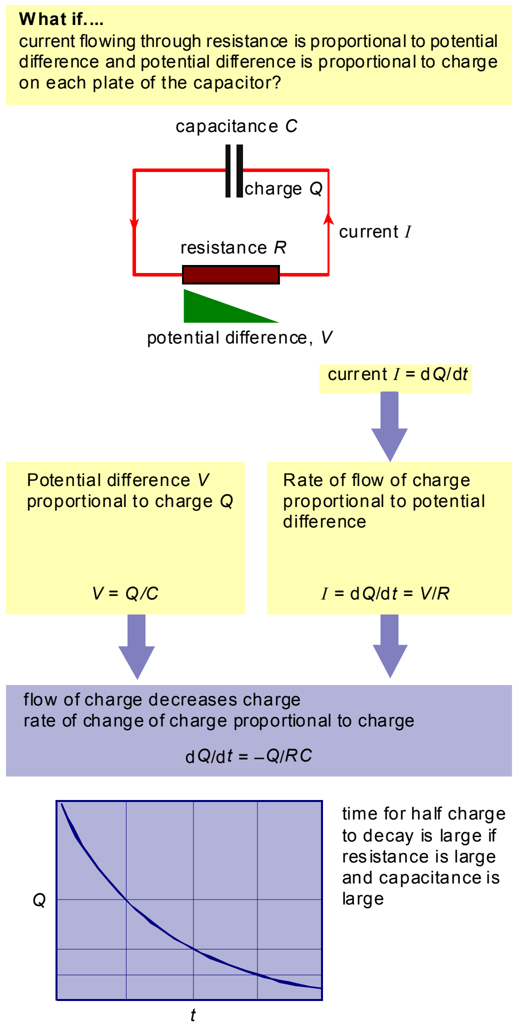
An exponential change arises because the rate of loss of water is proportional to the amount of water left.



Charge running out

An exponential change arises because the rate of loss of charge is proportional to the amount of charge left.

Exponential decay of charge



Charge decays exponentially if the current is proportional to potential difference, and the capacitance *C* is constant

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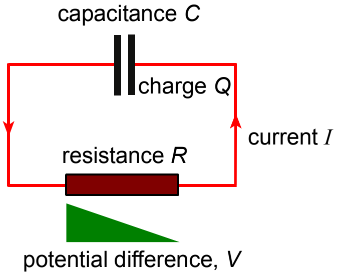
Exponential decay of charge

Exponential decay of charge

What if....
current flowing through resistance is proportional to potential difference and potential difference is proportional to charge on capacitor?



current $I = dQ/dt$

Potential difference V proportional to charge Q

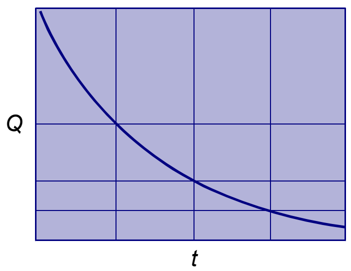
$V = Q/C$

Rate of flow of charge proportional to potential difference

$I = dQ/dt = V/R$

flow of charge decreases charge
rate of change of charge proportional to charge

$dQ/dt = -Q/RC$



time for half charge to decay is large if resistance is large and capacitance is large

Charge decays exponentially if current is proportional to potential difference, and capacitance *C* is constant

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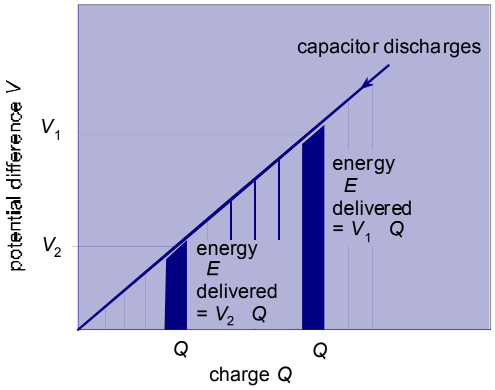
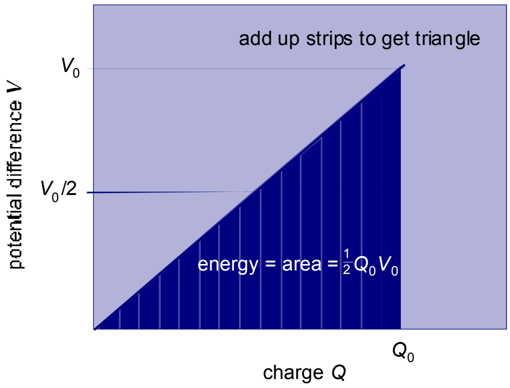
Energy stored by a capacitor

Energy stored by a capacitor





Energy delivered at p.d. *V* when a small charge *Q* flows *E* = *V* *Q*

Energy *E* delivered by same charge *Q* falls as *V* falls

Energy delivered = charge average p.d.

Energy delivered = $\frac{1}{2} Q_0 V_0$

Capacitance, charge and p.d.

$C = Q/V$ → $Q_0 = CV_0$, $V_0 = Q_0/C$

Equations for energy stored

$E = \frac{1}{2} Q_0 V_0$

$E = \frac{1}{2} C V_0^2$

$E = \frac{1}{2} Q_0^2 / C$

The energy stored by a capacitor is $\frac{1}{2}QV$

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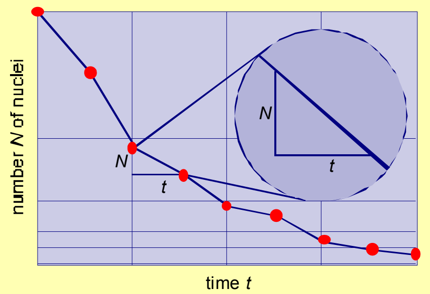
Smoothed out radioactive decay

Smoothed-out radioactive decay

Actual, random decay



probability p of decay in short time t is proportional to t:

$p =$ t

average number of decays in time t is pN

t short so that N much less than N

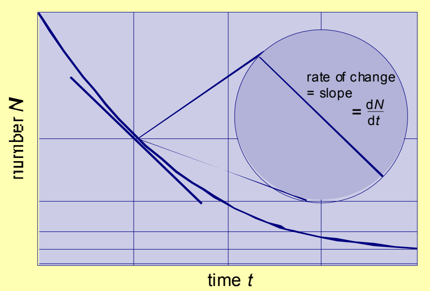
change in $N =$ $N =$ –number of decays

$N = -pN$

$N = -$ N t

$$\frac{N}{t} = - N$$

Simplified, smooth decay



Consider only the smooth form of the average behaviour.
In an interval dt as small as you please:

probability of decay $p =$ dt

number of decays in time dt is pN

change in $N =$ dN = –number of decays

d$N = -pN$

d$N = -$ N dt

$$\frac{dN}{dt} = - N$$

Actual, random decay fluctuates. The simplified model smooths out the fluctuations

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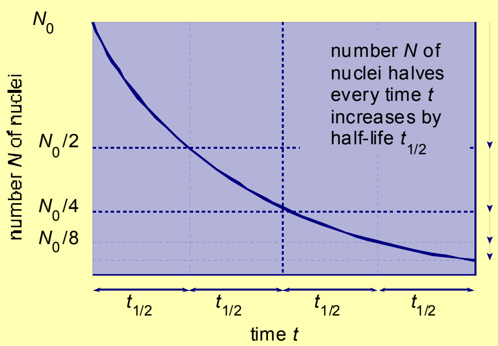
Radioactive decay used as a clock

Clocking radioactive decay

Half-life



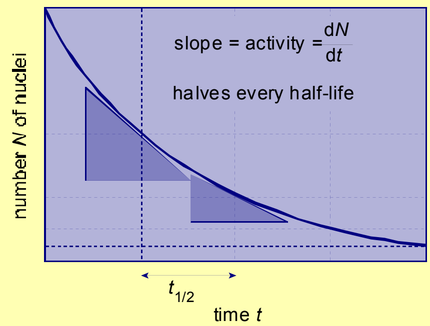
In any time t the number N is reduced by a constant factor

In one half-life $t_{1/2}$ the number N is reduced by a factor 2

In L half-lives the number N is reduced by a factor 2^L

(e.g. in 3 half-lives N is reduced by the factor $2^3 = 8$)

Activity



Radioactive clock

Measure activity. Activity proportional to number N left

Find factor F by which activity has been reduced

Calculate L so that $2^L = F$

$L = \log_2 F$

age = $t_{1/2} L$

The half-life of a radioactive isotope can be used as a clock

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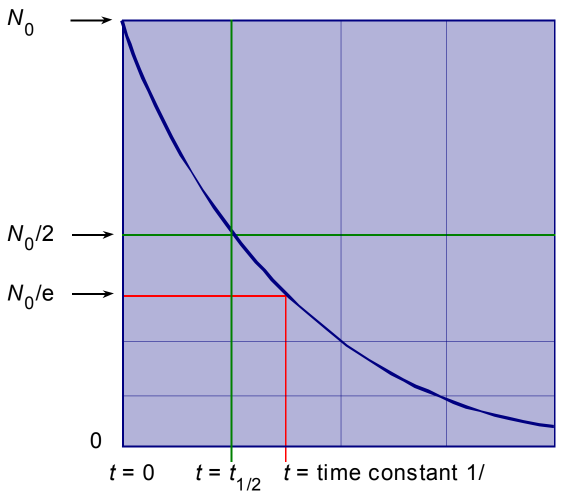
Half-life and time constant

Radioactive decay times

$dN/dt = -\ N$ $\qquad$ $N/N_0 = e^{-\ t}$



Time constant 1/
at time $t = 1/$
$N/N_0 = 1/e = 0.37$ approx.
$t = 1/$ is the time constant of the decay

Half-life $t_{1/2}$

at time $t_{1/2}$ number N becomes $N_0/2$

$N/N_0 = \frac{1}{2} = -\ \exp(-\ t_{1/2})$

$\ln \frac{1}{2} = -\ t_{1/2}$

$t_{1/2} = \frac{\ln 2}{} = \frac{0.693}{}$

$\ln 2 = \log_e 2$

The half-life $t_{1/2}$ is related to the decay constant

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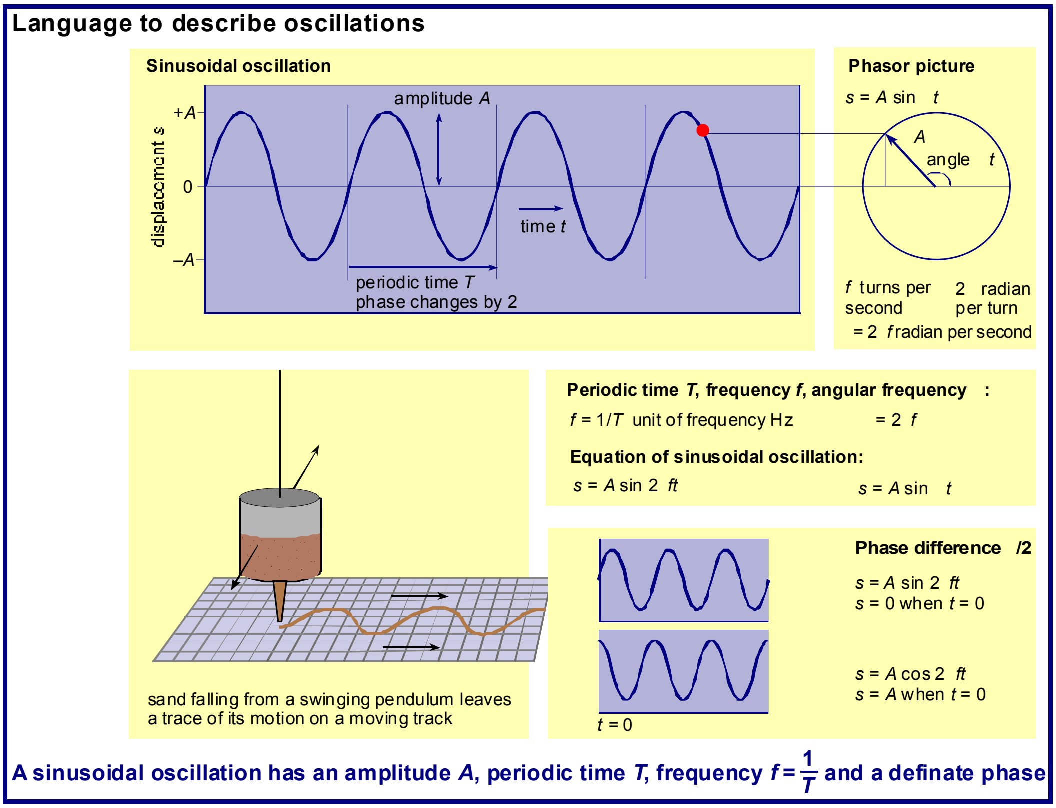
A language to describe oscillations



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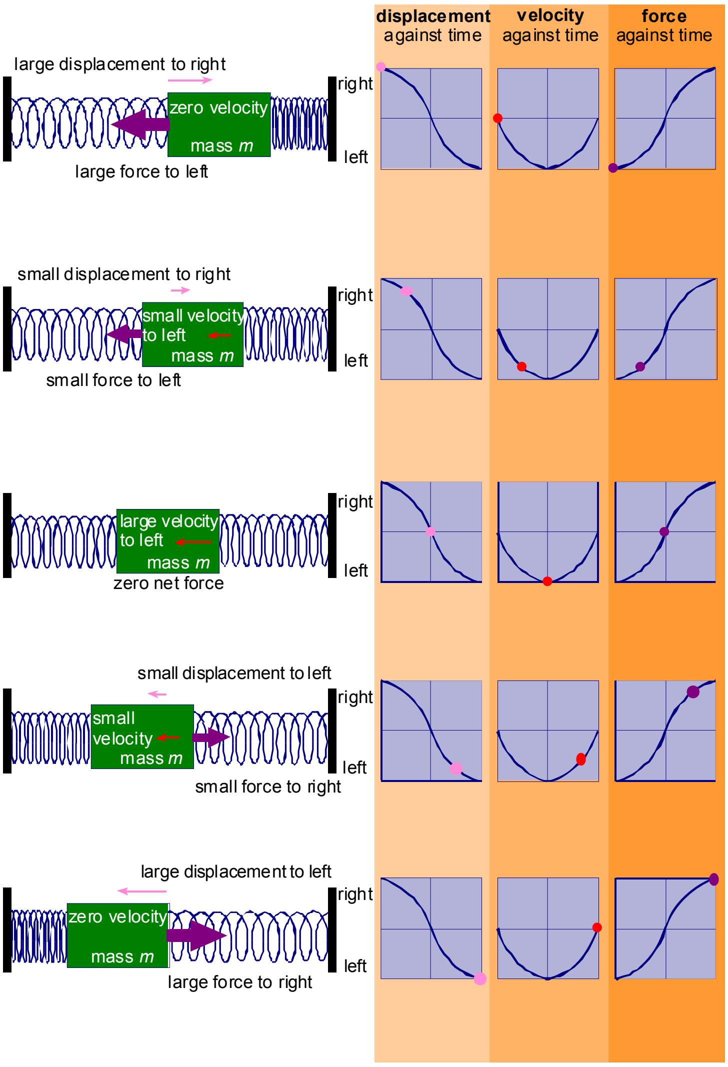
Snapshots of the motion of a simple harmonic oscillator

Motion of a harmonic oscillator



Everything about harmonic motion follows from the restoring force being proportional to minus the displacement

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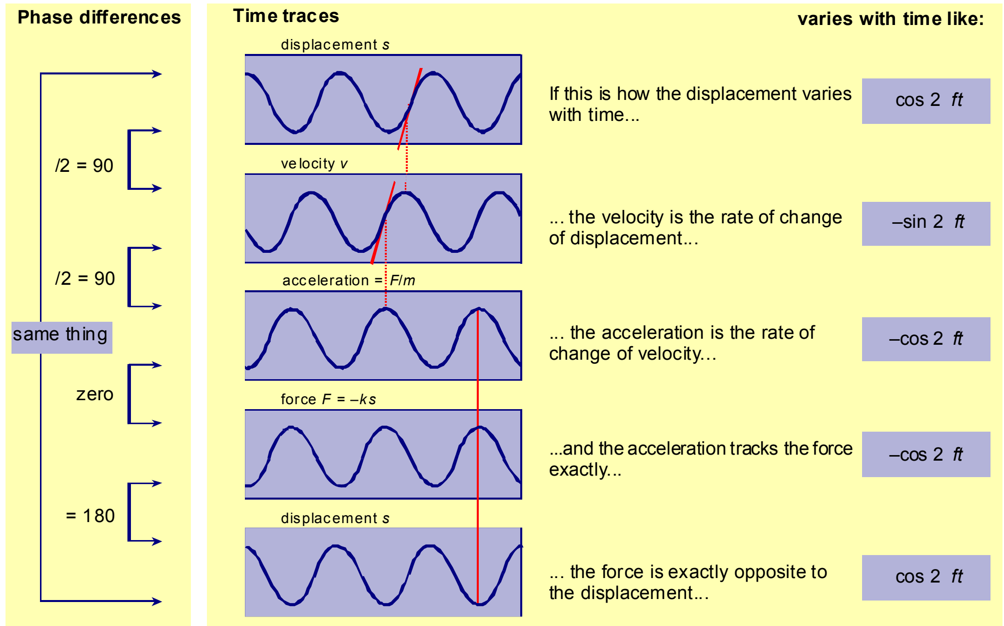
Graphs of simple harmonic motion

Force, acceleration, velocity and displacement



Graphs of displacement, velocity, acceleration and force against time have similar shapes but differ in phase

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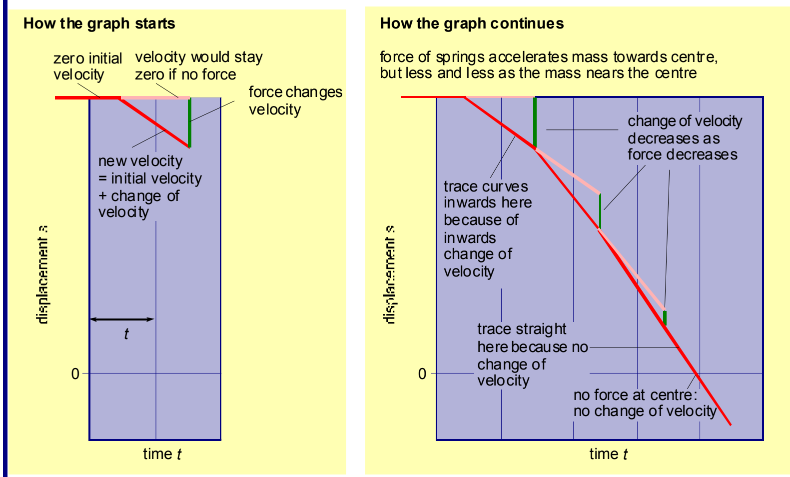
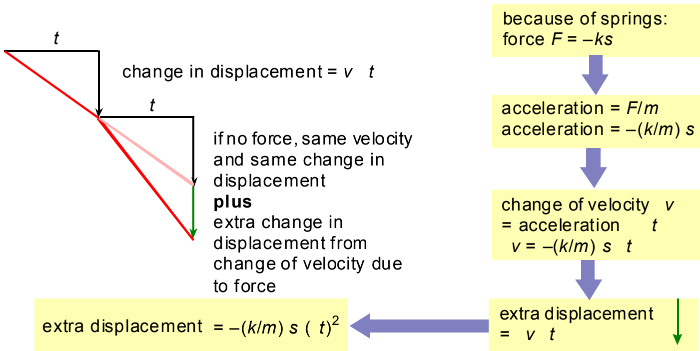
Step by step through the dynamics

Dynamics of a harmonic oscillator



Constructing the graph



Health warning! This simple (Euler) method has a flaw. It always changes the displacement by too much at each step. This means that the oscillator seems to gain energy!

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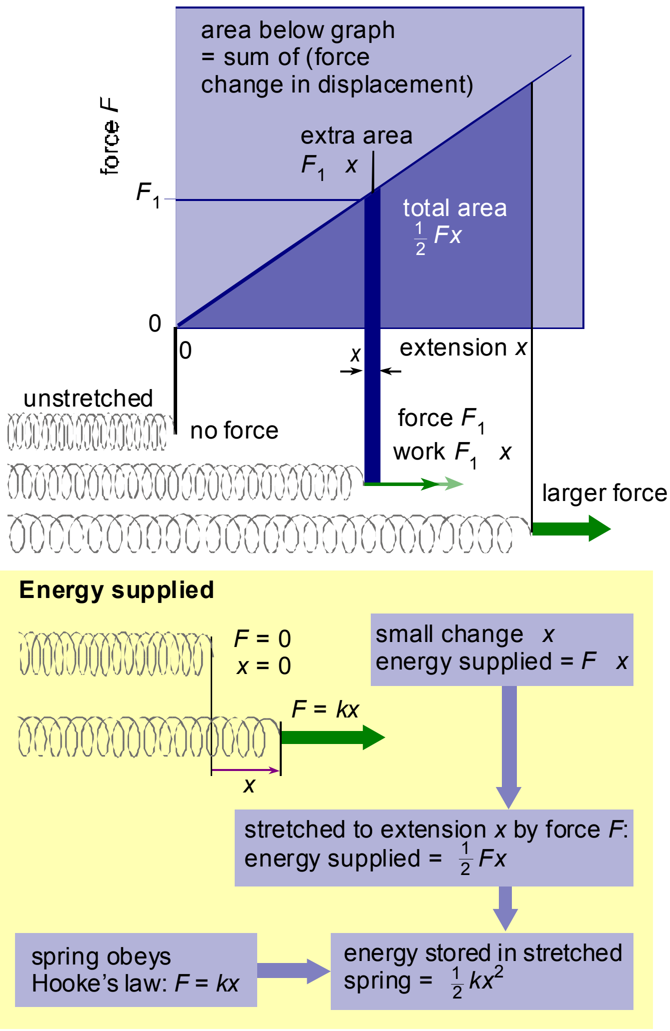
Elastic energy

The relationship between the force to extend a spring and the extension determines the energy stored.

Energy stored in a spring



Energy stored in a stretched spring is $\frac{1}{2}kx^2$

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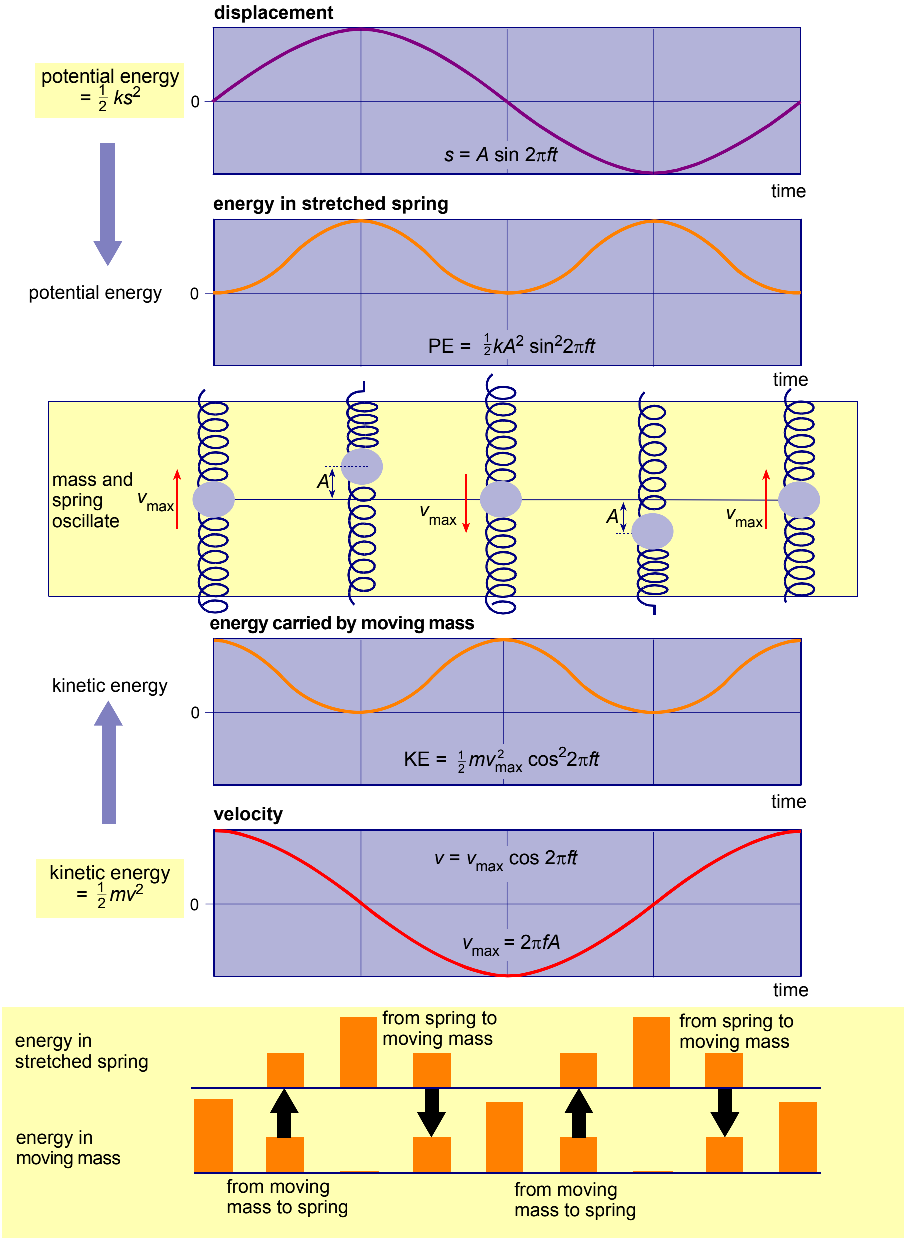
Energy flow in an oscillator

The energy sloshes back and forth between being stored in a spring and carried by the motion of the mass.

Energy flow in an oscillator



The energy stored in an oscillator goes back and forth between stretched spring and moving mass, between potential and kinetic energy

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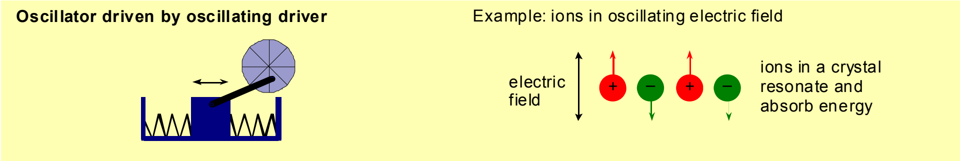
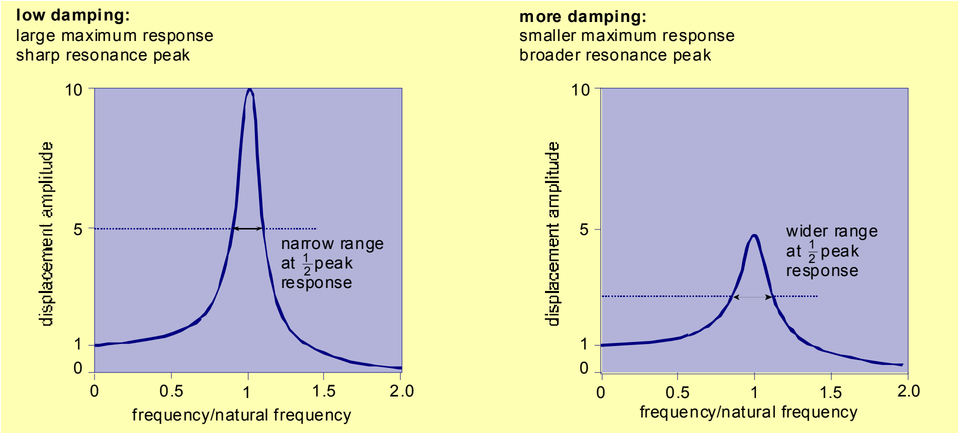
Resonance

Resonance occurs when driving frequency is equal to natural frequency. The amplitude at resonance, and just away from resonance, is affected by the damping.

Resonant response





Resonant response is at maximum when the frequency of a driver is equal to the natural frequency of an oscillator

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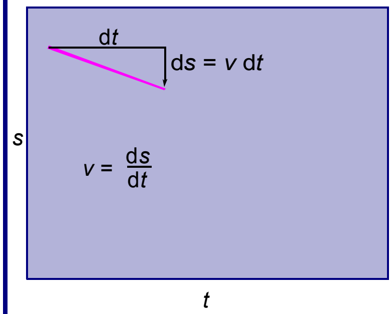
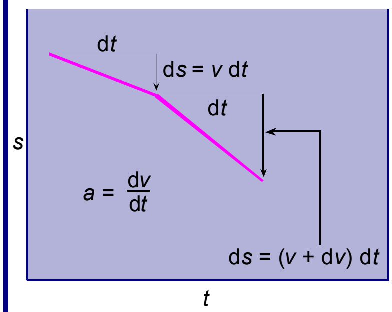
Rates of change

Changing rates of change



slope = rate of change of displacement
= velocity v

rate of change of slope = rate of change of velocity

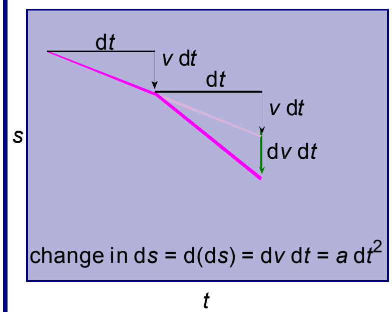


new slope = new rate of change of displacement

= new velocity (v + dv)

new ds = (v + dv) dt

dv = a dt



change in ds = d(ds) = dv dt
= a dt^2

$$\frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2} = a$$

The first derivative ds/dt says how steeply the graph slopes.
The second derivative d^2s/dt^2 says how rapidly the slope changes

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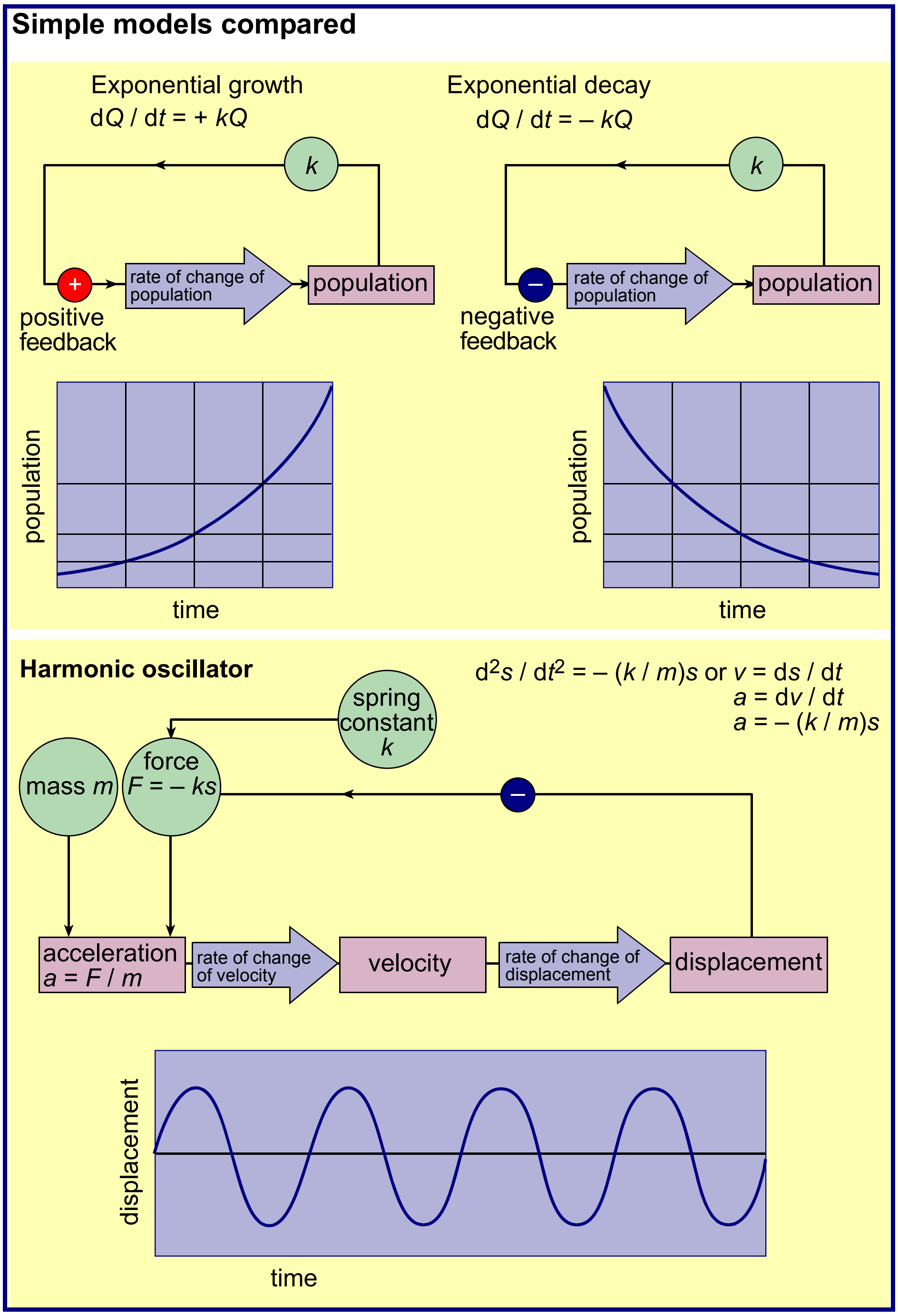
Comparing models



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